ISRAEL JOURNAL OF MATHEMATICS **162** (2007), 221–233
DOI: 10.1007/s11856-007-0096-5

ALGEBRAIC CONVERGENCE THEOREMS OF n-DIMENSIONAL KLEINIAN GROUPS

BY

Xiantao Wang*

Department of Mathematics, Hunan Normal University
Changsha, Hunan 410081, People's Republic of China
e-mail: xtwang@hunnu.edu.cn

ABSTRACT

Let $\{G_{r,i}\}$ be a sequence of r-generator n-dimensional Kleinian groups and G_r be its algebraic limit group. In this paper, we prove that G_r is a Kleinian group if $\{G_{r,i}\}$ satisfies some conditions. Our results are generalizations of the corresponding known ones.

1. Introduction

In this paper, we will adopt the same notations as in [1, 5, 17] such as the n-dimensional sense-preserving Möbius group $M(\bar{R}^n)$, the n-dimensional Clifford group Γ_n and the n-dimensional Clifford matrix group $SL(2,\Gamma_n)$.

Let

$$PSL(2,\Gamma_n) = SL(2,\Gamma_n)/\{\pm I\},$$

where I denotes the identity matrix.

By [1], $PSL(2,\Gamma_n)$ is isomorphic to $M(\bar{R}^n)$. We will identify the element in $M(\bar{R}^n)$ with its representation in $PSL(2,\Gamma_n)$.

In the following, I also denotes the identity mapping in $M(\bar{R}^n)$.

A subgroup G of $M(\bar{R}^n)$ is called **elementary** if it has a finite G-orbit in $\bar{H}^{n+1}(= H^{n+1} \cup \bar{R}^n)$ (cf., [5]). Otherwise, we will call G **non-elementary**.

In this paper, a **Kleinian** group means a non-elementary and discrete subgroup in $M(\bar{R}^n)$.

---

* The research was partly supported by Program for NCET (No. 04-0783) and NSFs of China (No. 10771059, No. 10571048) and Hunan (No. 05JJ10001).
Received December 11, 2005

Let $\{G_i\}$ be a sequence of subgroups in $M(\bar{R}^n)$ and each generated by $g_{1,i}, g_{2,i}, \ldots, g_{r,i}$, where $r = 1, 2, \ldots$. If for each $1 \le t \le r$,

$$g_{t,i} \to g_t \in M(\bar{R}^n) \quad \text{as } i \to \infty,$$

then we call that $\{G_i\}$ algebraically converges to $G = \langle g_1, g_2, \ldots, g_r \rangle$ and G is the algebraic limit group of r-generator groups $\{G_i\}$. In order to emphasize r, $1 \le r \le \infty$, we always replace G_i and G by $G_{r,i}$ and G_r, respectively. If for each i, $G_{r,i}$ is a Kleinian group, the problem of when G_r is still a Kleinian group was investigated by a number of authors.

When $n = 2$ and $r < \infty$, Jørgensen and Klein proved ([12]) the following.

Theorem JK: *If each $G_{r,i}$ is a r-generator Kleinian group, then the algebraic limit group G_r is also a Kleinian group.*

It easily follows from the examples in [16] or [17] that Theorem JK could not be extended to n-dimensional case ($n \ge 3$) without any modifications. Under what condition(s) can one get an analogue of Theorem JK in $M(\bar{R}^n)$ when $n \ge 3$? This problem has been discussed by several authors.

When $r < \infty$, as one of the main results in [13], Martin proved

Theorem M: *Let G_r be the algebraic limit group of a sequence of r-generator Kleinian groups of $M(\bar{R}^n)$ of uniformly bounded torsion. Then G_r is a Kleinian group.*

Here a subset X of $M(\bar{R}^n)$ is said to have uniformly bounded torsion if there exists a positive number K with the following property:

$$\text{if } f \in X, \text{ then } \operatorname{ord}(f) \le K \text{ or } \operatorname{ord}(f) = \infty.$$

It easily follows from Theorem M and its proof that

Corollary 1.1: *If each $G_{r,i}$ is a torsion-free Kleinian group, then G_r is also a torsion-free Kleinian group.*

See [21] for further discussions about Corollary 1.1.

When $r \le \infty$, Apanasov ([2] or [3, 4]) proved

Theorem A: *Let $r \le \infty$. If the generator system $\{g_{t,i}\}_{t=1}^{r}$ of $G_{r,i}$ satisfies that none are elliptic and no two have any fixed point in common, and if all $G_{r,i}$ are Kleinian groups, then for each t ($1 \le t \le r$), $g_t = \lim_{i\to\infty} g_{t,i}$ is different from I.*

Obviously, when $n = 2$, Theorem M does not coincide with the classical result, Theorem JK. In [13], Martin pointed out that it is worthwhile remarking how one may weaken the condition "uniformly bounded torsion" in Theorem M.

In this paper, we consider the mentioned problem further. Our main results are

THEOREM 1.1: *Let $r < \infty$ and G_r be the algebraic limit group of a sequence of r-generator Kleinian groups $\{G_{r,i}\}$ of $M(\bar{R}^n)$. If $\{G_{r,i}\}$ satisfies EP-conditon (see Section 2 for the definition), then G_r is a Kleinian group.*

THEOREM 1.2: *Let $r \leq \infty$. If the generator system $\{g_{t,i}\}_{t=1}^{r}$ of $G_{r,i}$ satisfies that none are elliptic and no two have any fixed point in common, and if all $G_{r,i}$ are Kleinian groups, then*

(1) *all the generators $g_t = \lim_{i\to\infty} g_{t,i}$ are neither elliptic, nor fixed-point-free, nor I;*
(2) *G_r is a Kleinian group if G_r is non-elementary and $WY(G_r)$ (see Section 2 for the definition) is finite.*

Remark 1.1: By Proposition 2.3 in Section 2, we know that Theorem 1.1 is a generalization of Theorem M when $n \geq 3$. When $n = 2$, Proposition 2.2 implies that Theorem 1.1 completely coincides with Theorem JK. Theorem 1.2 is a further discussion of Theorem A.

2. Preliminaries

For $f \in M(\bar{R}^n)$, let $\tilde{f}$ denote the Poincaré extension of f (cf., [5]),

$$fix(f) = \{x \in \bar{R}^n : f(x) = x\}, \quad fix(\tilde{f}) = \{z \in H^{n+1} : \tilde{f}(z) = z\}$$

and for a set M, let $card(M)$ denote its cardinality.

Now, we give a classification to the elements of $M(\bar{R}^n)$ as follows.

Non-trivial element $f \in M(\bar{R}^n)$ is called

(1) **fixed-point-free** if $card[fix(f)] = 0$;
(2) **loxodromic** if $card[fix(f)] > 0$ and f can be conjugate in $SL(2,\Gamma_n)$ to $\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix}$, where $r > 0$, $r \neq 1$, $\lambda \in \Gamma_n$ and $|\lambda| = 1$;
(3) **parabolic** if $card[fix(f)] > 0$ and f can be conjugate in $SL(2,\Gamma_n)$ to $\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}$, where $a, b \in \Gamma_n$, $|a| = 1$, $b \neq 0$ and $ab = ba'$;
(4) **elliptic** if $card[fix(f)] > 0$ and f can be conjugate in $SL(2,\Gamma_n)$ to $\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}$, where $u \in \Gamma_n$, $|u| = 1$ and $u \notin R$.

PROPOSITION 2.1:

(1) *$f \in M(\bar{R}^n)$ is fixed-point-free if and only if $card[fix(\tilde{f})] = 1$. $M(\bar{R}^n)$ contains a fixed-point-free element if and only if n is odd and $n \geq 3$;*

(2) *$f(\neq I)$ is elliptic if and only if $card[fix(\tilde{f})] = \infty$.*

Now we introduce the following conditions. Let $\{G_i\}$ be a sequence of subgroups of $M(\bar{R}^n)$.

Property A: We say that $\{G_i\}$ satisfies **Property A** if the set $\{G_i\}$ contains no sequence $\{\langle f_{i_k}, g_{i_k}\rangle\}$ which satisfies that both f_{i_k}, $g_{i_k} \in G_{i_k}(\in \{G_i\})$ are elliptic and

$$fix(f_{i_k}) \cap fix(g_{i_k}) = \emptyset, \quad card(fix(f_{i_k})) = card(fix(g_{i_k})) = 2,$$
$$f_{i_k} \to I \quad \text{and} \quad g_{i_k} \to I \quad \text{as } k \to \infty.$$

E-condition: We say that a sequence $\{G_i\}$ satisfies **E-condition** if any sequence $\{f_{i_k}\}$ ($f_{i_k} \in G_{i_k}(\in \{G_i\})$) satisfying that for each k, $card[fix(f_{i_k})] = \infty$ and $f_{i_k} \to I$ as $k \to \infty$ has uniformly bounded torsion.

EP-condition: We say that $\{G_i\}$ satisfies **EP-condition** if the following are satisfied:

(1) for any sequence $\{f_{i_k}\}$, $f_{i_k} \in G_{i_k}(\in \{G_i\})$, if $card(fix(f_{i_k})) = \infty$, for each k, and $f_{i_k} \to f$ as $k \to \infty$, where f is I or parabolic, then $\{f_{i_k}\}$ has uniformly bounded torsion;

(2) $\{G_i\}$ satisfies Property A.

Jørgensen's inequality ([11]) implies that

PROPOSITION 2.2: *For $n = 1$ or 2, each sequence $\{G_i\}$ of discrete groups of $M(\bar{R}^n)$ satisfies Property A. Hence it also satisfies E-condition and EP-condition.*

The following propositions are obvious.

PROPOSITION 2.3: *For any sequence $\{G_i\}$ of discrete groups of $M(\bar{R}^n)$, if $\{G_i\}$ has uniformly bounded torsion, in particular, if $\{G_i\}$ is torsion-free, then it satisfies both E-condition and EP-condition.*

PROPOSITION 2.4: *A sequence $\{G_i\}$ of discrete groups of $M(\bar{R}^n)$ satisfies E-condition if and only if it satisfies* **Condition A** *in* [8].

Remark 2.1: The example in [6] or [8] shows that there exists a sequence of discrete groups of $M(\bar{R}^n)$ which satisfies both E-condition and EP-condition, but it is neither torsion-free nor has uniformly bounded torsion.

The following lemma is crucial for us (Theorem 11 in [20]).

Lemma 2.1: *Let f, $g \in M(\bar{R}^n)$. If the group $\langle f, g\rangle$ generated by f and g is a Kleinian group, then*

$$\|f - I\| \cdot \|g - I\| \geq 1/32.$$

Let $G \in M(\bar{R}^n)$ be non-elementary and let $\mathbf{H(G)}$ denote the set consisting of all loxodromic elements in G. As in [16] or [17], we let

$$WY(G) = \{g \in G : fix(f) \subset fix(g) \text{ for all } f \in H(G)\}.$$

Obviously, for any non-elementary subgroup G of $M(\bar{R}^n)$, each element in $WY(G)$ fixes $L(G)$ (the limit set of G) pointwise. By using $WY(G)$, we have obtained several criteria for subgroups of $M(\bar{R}^n)$ to be discrete, see [16, 17, 18].

Remark 2.2: When $n = 1$ or 2, for any non-elementary group G in $M(\bar{R}^n)$, $WY(G) = \{I\}$.

The next result follows directly from [10, 15] or [19].

Lemma 2.2: *If $G \in M(\bar{R}^n)$ is discrete and each non-trivial element of G is of finite order, then G is finite.*

Lemma 2.3 ([14]): *If $G \subset M(\bar{R}^n)$ is finite, then* $\mathrm{card}(\bigcap_{f \in G} fix(f)) \neq 1$.

3. Main Lemma

Let $\{f_i\}$ and $\{g_i\}$ be two sequences of $M(\bar{R}^n)$, which converge to f and g, respectively, and for each i, $\langle f_i, g_i\rangle$ is a Kleinian group. The question of when the group $\langle f, g\rangle$ is a Kleinian group has been discussed by several authors, see [7, 8, 21] etc. The main goal of this section is to prove a result which is a generalization of the corresponding questions in [7, 8, 21] by using different methods.

In order to prove our main lemma, we need the following.

Lemma 3.1: *Let $f \in M(\bar{R}^n)$ be parabolic. If the group $\langle f, g\rangle$ generated by f and $g \in M(\bar{R}^n)$ is non-elementary, then the elements f, gfg^{-1} and $(fg)f(fg)^{-1}$ have no common fixed point.*

Our main lemma is as follows.

Lemma 3.2: *Let $\{f_i\}$ and $\{g_i\}$ be as stated above. Suppose that each group $\langle f_i, g_i\rangle$ is a Kleinian group and each f_i is of infinite order. Then f is of infinite order and $\langle f, g\rangle$ is a Kleinian group if $\{\langle f_i, g_i\rangle\}$ satisfies* **E-condition**.

Proof:

(1) We first prove that $\langle f, g\rangle$ is discrete.

Suppose that $\langle f, g\rangle$ is not discrete. Then $\langle f, g\rangle$ is infinite and there is a sequence $\{h_j\}$ of $\langle f, g\rangle$ such that

$$h_j \to I \quad \text{as } j \to \infty.$$

Let $h_{j,i}$ be the corresponding elements in $\langle f_i, g_i\rangle$ such that

$$h_{j,i} \to h_j \quad \text{as } i \to \infty. \tag{3.1}$$

These imply that there is a sequence $\{h_{j_k,i_k}\}$ such that

$$h_{j_k,i_k} \in \langle f_{i_k}, g_{i_k}\rangle \quad \text{and} \quad h_{j_k,i_k} \to I \quad \text{as } k \to \infty.$$

Since $\{\langle f_i, g_i\rangle\}$ satisfies **E-condition**, we have

$$card[fix(h_{j_k,i_k})] \leq 2.$$

Suppose that there is a subsequence of $\{\langle f_i, g_i\rangle\}$ (denoted in the same way) such that each f_i is parabolic, then, by Lemma 3.1, the parabolic elements f_i, $g_i f_i g_i^{-1}$ and $(f_i g_i) f_i (f_i g_i)^{-1}$ have no common fixed point each other. By (3.1), we know, there is a positive M_1 such that for all $k > M_1$,

$$\|h_{ijk_p} - I\| \cdot \|h_{j_k,i_k} - I\| < 1/32 \quad p = 1, 2, 3,$$

where $h_{ijk_1} = f_{i_k}$, $h_{ijk_2} = g_{i_k} f_{i_k} g_{i_k}^{-1}$ and $h_{ijk_3} = (f_{i_k} g_{i_k}) f_{i_k} (f_{i_k} g_{i_k})^{-1}$.

Since $\langle h_{ijk_p}, h_{j_k,i_k}\rangle$ are discrete, we know that they are elementary by Lemma 2.1. These imply that $fix(h_{ijk_p}) \subset fix(h_{j_k,i_k})$ ($p = 1, 2, 3$). Then $card[fix(h_{j_k,i_k})] \geq 3$ since each f_{i_k} is parabolic. This is a contradiction.

Then we may assume that all f_i are loxodromic. Since each $\langle f_i, g_i\rangle$ is discrete and non-elementary, we know that

$$fix(f_i) \cap fix(g_i f_i g_i^{-1}) = \emptyset.$$

Similar discussions as above will lead to a contradiction.

The above shows that $\langle f, g\rangle$ is discrete.

(2) Then we prove that $\langle f, g\rangle$ is non-elementary.

We claim that f is of infinite order, i.e., f is parabolic or loxodromic since $\langle f, g\rangle$ is discrete. Suppose that there is a positive M such that $f^M = I$. Then $f_i^M \neq I$ and $f_i^M \to I$ as $i \to \infty$. Hence for sufficiently large i,

$$\|f_i^M - I\| \cdot \|g_i - I\| < 1/32.$$

By Lemma 2.1, we know that $\langle f_i^M, g_i\rangle$ are elementary for all $i > M$ since they are discrete. So are $\langle f_i, g_i\rangle$. It is a contradiction.

Now we come to prove that $\langle f, g\rangle$ is non-elementary.

Suppose that $\langle f, g\rangle$ is elementary. Since $\langle f, g\rangle$ is discrete, we can find two positive integers t and s such that

$$[f^t, gf^sg^{-1}] = I.$$

Let $h_i = [f_i^t, g_if_i^sg_i^{-1}]$. Then

$$h_i \in \langle f_i, g_i\rangle,\ h_i \neq I \quad \text{and} \quad h_i \to I \text{ as } i \to \infty.$$

Similar discussions as in the proof of part (1) imply that this is impossible. Hence $\langle f, g\rangle$ is non-elementary.

Remark 3.1: In [8], Fang and Nai proved that under the assumptions of Lemma 3.2, and if all g_i are not of finite order and $\{\langle f_i, g_i\rangle\}$ satisfies Condition A, i.e., E-Condition in our term, then f is parabolic or loxodromic and $\langle f, g\rangle$ is non-elementary. In the proof of $\langle f, g\rangle$ being non-elementary, Fang and Nai divided their arguments into two cases: (a): $\langle f, g\rangle$ is discrete and (b): $\langle f, g\rangle$ is not discrete. They had a long discussion to get a contradiction under the suppositions of $\langle f, g\rangle$ being elementary and non-discrete. The proof of Lemma 3.2 implies that the hypothesis "all g_i not being of finite order" is unnecessary and the case "$\langle f, g\rangle$ being non-discrete" cannot occur since we have proved that $\langle f, g\rangle$ is discrete.

4. The Proof of Theorem 1.1

Claim 1: *G_r is discrete.*

Suppose that G_r is not discrete. Then there is a sequence $\{g_j\}$ of G_r such that

$$g_j \to I \quad \text{as } j \to \infty.$$

It follows from the proof of Lemma 3.2 that we can find a sequence $\{g_{j_k,i_k}\}$ such that

$$g_{j_k,i_k} \in G_{r,i_k} \quad \text{and} \quad g_{j_k,i_k} \to I \quad \text{as } k \to \infty. \tag{4.1}$$

We divide our discussion into two cases in the following.

Case I: n is odd.

Since $\{G_{r,i}\}$ satisfies EP-condition, by Proposition 2.1, we may assume that for each k, g_{j_k,i_k} is of infinite order (i.e., g_{j_k,i_k} is parabolic or loxodromic), or fixed-point-free.

If the sequence $\{g_{j_k,i_k}\}$ contains a subsequence (denoted by the same way) such that all its elements are of infinite order, then for each k, there is at least one generator of G_{r,i_k}, say f_{1,i_k} (if needed , we can take a subsequence of $\{g_{j_k,i_k}\}$ since r is finite), such that $\langle f_{1,i_k}, g_{j_k,i_k}\rangle$ is non-elementary. By Lemma 2.1, it is impossible.

Hence, we may assume that each g_{j_k,i_k} in (4.1) is fixed-point-free and of order at least 3. Further, we may assume that g_{j_k,i_k} does not interchange any two different points. If not so, then

$$g^2_{j_k,i_k} \neq I, \quad card[fix(g^2_{j_k,i_k})] = \infty \quad \text{and} \quad g^2_{j_k,i_k} \to I \quad \text{as } k \to \infty.$$

This is impossible.

By (4.1), there is a positive number N such that for all $k > N$ and each generator f_{s,i_k} of $G_{r,i_k}(s = 1, 2, \ldots, r)$,

$$\|g_{j_k,i_k} - I\| \cdot \|f_{s,i_k} - I\| < 1/32.$$

This implies that $\langle g_{j_k,i_k}, f_{s,i_k}\rangle$ is elementary. Hence, each f_{s,i_k} is of finite order, and g_{j_k,i_k} and $f_{s,i_k}(s = 1, 2, \ldots, r)$ have one and only one common fixed point in H^{n+1}. Since g_{j_k,i_k} has only one fixed point in H^{n+1}, this implies that G_{r,i_k} is elementary. It is a contradiction.

Case II: n is even.

Since $\{G_{r,i}\}$ satisfies EP-condition, by Proposition 2.1, we may assume that

$$card[fix(g_{j_k,i_k})] = 1 \text{ or } 2$$

for each k.

Furthermore, similar discussions as in Case I show that we may assume that $ord(g_{j_k,i_k}) < +\infty$ and $card(fix(g_{j_k,i_k})) = 2$ for each k.

A: Suppose that there is a subsequence of $\{G_{r,i_k}\}$ (still denoted in the same way) such that for each k, at least one of the generators of G_{r,i_k}, say f_{1,i_k} (if necessary, passing to a subsequence), satisfies that

$$fix(g_{j_k,i_k}) \cap fix(f_{1,i_k} g_{j_k,i_k} f^{-1}_{1,i_k}) = \emptyset.$$

It is impossible since $\{G_{r,i}\}$ satisfies EP-condition and $f_{1,i_k} g_{j_k,i_k} f^{-1}_{1,i_k} \to I$ as $k \to \infty$.

Now we may assume that for each k and each generator f_{s,i_k} $(s = 1, 2 \ldots, r)$ of G_{r,i_k},

$$fix(g_{j_k,i_k}) \cap fix(f_{s,i_k} g_{j_k,i_k} f_{s,i_k}^{-1}) \neq \emptyset.$$

B: Suppose that there is a subsequence of $\{G_{r,i_k}\}$ (denoted in the same way) such that at least one of the generators of $\{G_{r,i_k}\}$, say f_{1,i_k} (if necessary, passing to a subsequence), satisfies that

$$card(fix(g_{j_k,i_k}) \cap fix(f_{1,i_k} g_{j_k,i_k} f_{1,i_k}^{-1})) = 1.$$

By Lemma 2.3, $\langle g_{j_k,i_k}, f_{1,i_k} g_{j_k,i_k} f_{1,i_k}^{-1}\rangle$ is infinite. It follows from Lemma 2.2 that $\langle g_{j_k,i_k}, f_{1,i_k} g_{j_k,i_k} f_{1,i_k}^{-1}\rangle$ contains parabolic element, say q_{i_k}. Then

$$q_{i_k} \in G_{r,i_k}, q_{i_k} \to I \quad \text{as } k \to \infty.$$

By the discussions as in the first part of Case I, we know that it is impossible.

Cases A and B imply that we may assume that for each k and each generator f_{s,i_k} $(s = 1, 2 \ldots, r)$ of G_{i_k}, $fix(g_{j_k,i_k}) = fix(f_{s,i_k} g_{j_k,i_k} f_{s,i_k}^{-1})$. Then G_{r,i_k} preserves $fix(g_{j_k,i_k})$ setwise. This means that G_{r,i_k} is elementary. It is our desired contradiction.

The above proves the discreteness of G_r.

Claim 2: *G_r is infinite.*

Suppose that G_r is finite. As in the proof of Proposition 5.8 in [13], we can find a sequence $\{h_i\}$ such that $\{h_i\} \in G_{r,i}$ and $h_i \to I$ as $i \to \infty$. Similar discussions as in the proof of Claim 1 show that it is impossible. Hence G_r is infinite.

Claim 3: *G_r is non-elementary.*

Suppose that G_r is elementary. It follows from Claim 2 and Lemma 2.2 that G_r contains some element h of infinite order, i.e, h is parabolic or loxodromic. Let $\{h_i\}$ be the corresponding words in $\{G_{r,i}\}$. Then it is clear that

$$h_i \to h \quad \text{as } i \to \infty.$$

Suppose that h is loxodromic. Then for all sufficiently large i, h_i are loxodromic. For each generator f_s $(s = 1, 2 \ldots, r)$ of G_r, there are k_s and p_s such that $[h^{k_s}, f_s h^{p_s} f_s^{-1}] = I$ (cf., [9]). Then there exists some $1 \leq s \leq r$, passing to a subsequence if needed, such that

$$[h_i^{k_s}, f_{s,i} h_i^{p_s} f_{s,i}^{-1}] \neq I \quad \text{and} \quad [h_i^{k_s}, f_{s,i} h_i^{p_s} f_{s,i}^{-1}] \to I \quad \text{as } i \to \infty.$$

The discussions as in Claim 1 show that it is impossible. It follows that h is parabolic.

Since $\{G_{r,i}\}$ satisfies EP-condition, we know that the order of h_i is infinite (i.e., h_i is parabolic or loxodromic) or fixed-point-free or elliptic element with only two fixed points. Suppose that there is a subsequence of $\{h_i\}$ such that each h_i is parabolic or loxodromic. Then for i, there is a generator, say $f_{1,i}$, such that the group $\langle f_{1,i}, h_i\rangle$ is non-elementary. By Lemma 3.2, the limit group of the sequence $\{\langle f_{1,i}, h_i\rangle\}$ is non-elementary. This implies that G_r is non-elementary. This is a contradiction.

Suppose that all h_i are fixed-point-free.

For any fixed $s(s = 1, 2, \ldots, r)$, let $f_{s,i}$ be the corresponding word of f_s in $G_{r,i}$. Then $f_s h f_s^{-1}$ is parabolic and $f_{s,i} h_i f_{s,i}^{-1}$ are fixed-point-free. Thus there exist two natural numbers k and t such that

$$[h^k, f_s h^t f_s^{-1}] = I \quad \text{(cf., [13])}.$$

It follows from the above discussions that we may assume that for all i, $h_i^k \neq I$, $h_i^t \neq I$ and they are fixed-point-free.

If $[h_i^k, f_{s,i} h_i^t f_{s,i}^{-1}] = I$, then h_i^k and $f_{s,i}$ have one and only one common fixed point in H^{n+1}. This implies that there is at least one generator of G_i, say $f_{1,i}$, such that $[h_i^k, f_{1,i} h_i^t f_{1,i}^{-1}] \neq I$, otherwise, $G_{r,i}$ is elementary. We may assume that there is a sequence, say $\{f_{s,i}\}$, since r is finite, such that

$$[h_i^k, f_{s,i} h_i^t f_{s,i}^{-1}] \neq I \quad \text{and} \quad [h_i^k, f_{s,i} h_i^t f_{s,i}^{-1}] \to I \quad \text{as } i \to \infty.$$

By the proof of Claim 1, we know that it is impossible.

Hence we may assume that each h_i is elliptic with only two fixed points. Furthermore, we may assume that for each fixed number m, all h_i^m are elliptic with only two fixed points and do not interchange two different points, since $\{G_{r,i}\}$ satisfies EP-condition.

Similar discussions as above show that it is impossible.

The proof is completed.

5. The Proof of Theorem 1.2

The proof of (1): It follows from Theorem A or Lemma 2.1 that all g_s are not I.

Suppose g_s is elliptic or fixed-point-free for some $1 \leq s \leq r$. Without loss of generality, we may assume that $s = 1$. If g_1 has order k, then

$$g_{1,i}^k \to g_1^k = I.$$

This is impossible by Theorem A. If g_1 has the order of infinity, then there is a sequence $\{k\}$ such that

$$g_1^k \to I.$$

Then for sufficiently large k,

$$\|g_1^k - I\| \cdot \|g_2 - I\| < 1/32.$$

It follows that, for large enough i,

$$\|g_{1,i}^k - I\| \cdot \|g_{2,i} - I\| < 1/32.$$

This implies that $fix(g_{1,i}) = fix(g_{2,i})$. This is impossible.

The proof of (2): Suppose G_r is not discrete. Then there is a sequence $\{f_t\} \subset G_r$ such that

$$f_t \to I \quad \text{as } t \to \infty.$$

As G_r is non-elementary, we know that it contains at least two loxodromic elements h_1 and h_2 such that $fix(h_1) \cap fix(h_2) = \emptyset$. Then for large enough t,

$$\|f_t - I\| \cdot \|h_j - I\| < 1/32,$$

for $j = 1, 2$. Let $h_{j,i}$ be the corresponding element of h_j in $G_{r,i}$ $(j = 1, 2)$. Then for sufficiently large i,

$$\|f_{t,i} - I\| \cdot \|h_{j,i} - I\| < 1/32,$$

for $j = 1, 2$. Lemma 2.1 implies that $\langle f_{t,i}, h_{j,i} \rangle$ is elementary $(j = 1, 2)$. It follows that for large enough i,

$$fix(h_{j,i}) \subset fix(f_{t,i}).$$

Hence, there is $T > 0$ such that for all $t \geq T$,

$$fix(h_j) \subset fix(f_t)$$

for $j = 1, 2$.

Let

$$F_{T'} = \bigcap_{t \geq T'} fix(f_t).$$

Then

$$1 \leq \dim(F_{T'}) = r \leq n - 2.$$

Suppose that G_r contains a loxodromic element h such that $fix(h) \cap F_{T'} \neq fix(h)$. Then the discussions as above shows that there is $T_1 > T'$ such that

$$fix(h) \subset F_{T_1}$$

and

$$r + 1 \leq \dim(F_{T_1}) \leq n - 2.$$

Thus we know that there exists $T(\geq T_1)$ such that for any loxodromic element $g \in G_r$,

$$fix(g) \subset F_T.$$

This shows that

$$f_t \in WY(G_r)$$

for all $t \geq T$. This is the desired contradiction.

ACKNOWLEDGEMENT: The author thanks the referee for his (or her) valuable comments.

References

[1] L. V. Ahlfors, *On the fixed points of Möbius transformations in $\bar{R}^n$*, Annales Academiae Scientiarum Fennicae. Series A I. Mathematica Dissertationes **10** (1985), 15–27.

[2] B. N. Apanasov, *A universal property of Kleinian groups in the hyperbolic metric,* Soviet Mathematics Doklady **16** (1975), 1418–1421.

[3] B. N. Apanasov, *Discrete Groups in Space and Uniformization Problems,* Mathematics and its Applications (Soviet Series), vol. 40, Kluwer Academic Publishers Group, Dordrecht, 1991.

[4] B. N. Apanasov, *Conformal Geometry of Discrete Groups and Manifolds,* Walter de Gruyter, Berlin, New York, 2000.

[5] A. F. Beardon, *The Geometry of Discrete Groups,* Graduate text in Mathematics, vol. 91, Springer, 1983.

[6] A. F. Beardon and P. L. Waterman, *Strongly discrete subgroups of SL(2,C),* Journal of the London Mathematical Society **24** (1981), 325–328.

[7] V. Chuckrow, *On Schotty groups with application to Kleinian groups,* Annals of Mathematics **88** (1968), 47–61.

[8] A. Fang and B. Nai, *On the discreteness and convergence in n-dimensional Möbius groups,* Journal of the London Mathematical Society **61** (2000), 761–773.

[9] F. W. Gehring and G. J. Martin, *Discrete quasiconformal groups,* Proceedings of the London Mathematical Society **55** (1987), 331–385.

[10] L. Greenberg, *Discrete subgroups of the Lorentz group,* Mathematica Scandinavica **10** (1962), 85–107.

[11] T. Jørgensen, *On discrete groups of Möbius transformations,* American Journal of Mathematics **98** (1976), 739–749.

[12] T. Jørgensen and P. Klein, *Algebraic convergence of finitely generated Kleinian groups,* The Quarterly Journal of Mathematics. Oxford **33** (1982), 325–332.

[13] G. J. Martin, *On discrete Möbius groups in all dimensions,* Acta Mathematica **163** (1989), 253–289.

[14] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds,* Springer-Verlag, Berlin, 1994.

[15] A. Selberg, *On discontinuous groups in higher-dimensional symmetric spaces, Contributions to Function Theory*, Tata, 1960, pp. 147–164.

[16] X. Wang, *Dense subgroups of n-dimensional Möbius groups,* Mathematische Zeitschrift **243** (2003), 643–651.

[17] X. Wang and W. Yang, *Discreteness criteria of Möbius groups of high dimensions and convergence theorem of Kleinian groups,* Advances in Mathematics **159** (2001), 68–82.

[18] X. Wang, L. Liu and W. Cao, *Discreteness criteria for Möbius groups acting on* $\bar{R}^n$*,* Israel Journal of Mathematics **150** (2005), 357–368.

[19] P. Waterman, *Purely elliptic Möbius groups, Holomorphic Functions and Moduli II* (D. Drasin, ed.), Springer-Verlag, Berlin, 1988, pp. 173–178.

[20] P. Waterman, *Möbius transformations in several dimensions,* Advances in Mathematics **101** (1993), 87–113.

[21] N. J. Wielenberg, *Discrete Möbius groups: Fundamental polyhedra and convergence,* American Journal of Mathematics **99** (1977), 861–877.